Continuous simulation, differential inclusions and traveling in time

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SUMMARY
A simulation of a model with an ideal predictor is presented. The problem is equivalent to the problem of passing the information from the future to the present, or traveling into the past to use the present information and change the model trajectory. It is shown that the uncertainty over the future can be simulated using differential inclusions. A differential inclusion solver is described. An example of continuous simulation with the ideal predictor element is shown.

INTRODUCTION
Traveling in time has always been a fascinating problem exciting the imagination of science fiction writers, scientists, and normal people. There are well known paradoxes that arise when we admit the possibility of time traveling. What happens if someone travels backward in time and kills his grandfather? The same paradox results if someone travels forward in time, say 30 years, learns that his son is unhappy or bad person, comes back and decides to have no children at all?

There is a huge literature on time traveling, mostly in the field of science fiction. On the other hand, the relativistic theory provides new insight on the concept of time, and might permit us in the future to manipulate the time due to our needs. Recall that, according to the relativistic laws, the one-way trip to the future is possible. Unfortunately, it would still be difficult to go back. As for now, we cannot see what exactly will happens, and our possibilities to see what had happens in the past are quite limited.
Let us fokus on a simulated instead of the real word. Computer simulation permits us to play with models, treating the model time as merely one of the model variables. In continuous simulation we can integrate the model equations going forward or backward, simply changing the sign of the time increments. This, however, is not sufficient to simulate the time travel problem. Our problem statement is as follows. Given a deterministic continuous model with fixed initial conditions at time t_0, suppose that at each time instant t_x we have complete informationon on the past model trajectory relative to t_x, and no information on the future trajectory shape (for $t>t_x$). Now consider two cases:
1. Backward feedback. Suppose that our model is causal, in the sense of time. This means that the actual model state only depends on the actual and past inputs and on the past model trajectory. A simple example of such model is a system with time-delay elements that takes one or more past system states to determine the actual slope of the model trajectory. To simulate such system we need the trajectory to be stored in computer memory and used while advancing model time, which can be done without problems.
2. Forward feedback. Simply changing the sign of the time delay, replacing the delays with predictors we face a model of "forward time traveling". In other words, to determine the

change of the actual state, we need the information on the future states. The problem is that no such information exists. Note that this has nothing to do with algorithms like "corrector-predictor" method and time series projections. Any projection we could use is charged with some uncertainty, that means that our model is no longer deterministic. Moreover, using a result of a projection in the model equations may result in a trajectory completely different from the prediction which means that the trajectory we get does not satisfy the model equations (the use of false information).

Comparing the cases 1 and 2 we can see that the difference consists not only in the sign of the time delay. The question is if the case 2 can be simulated at all, and how to carry out the simulation.

The mathematical tool that can be used to simulate the above problem is known as **differential inclusion (DI)**, also known as differential equation with multivalued right hand side. The difference between an ordinary differential equation and a differential inclusion is that the right hand side of a DI is a set instead of a unique-valued function. The solution to a DI is also a set and not a single system trajectory. Note that the DIs have nothing to do with stochastic systems, where some uncertain variables are treated as random ones. The definition of a DI is completely deterministic.

To be more specific, let us note that the general form of a differential inclusion (DI) is as follows.

$$(1) \quad \begin{cases} dx/dt \in F(x,t) \\ x(0) \in I \\ where \quad x = (x_1, x_2, ..., x_n), \\ x \in X, \quad F \subset X, \quad t \in R \end{cases}$$

Here X is the *state space*, and I is the *initial set*, frequently reduced to one point *x(0)*. We suppose that the derivetaive of x belongs to the same tate space, and that the set F is bounded.

If the set f can be parametrized by an auxiliary variable u then the DI (1) can be given in the form of an *equivalent control system* (u being the control variable).

$$(2) \quad \begin{cases} dx/dt = f(x,u,t) \\ x(0) \in I \\ u \in C(x,t) \\ where \quad x = (x_1, x_2, ..., x_n) \\ f = (f_1, f_2, ..., f_n) \\ x \in X, \quad C \subset U, \quad t \in R \end{cases}$$

Here U denotes the *control space*. The relation between the set F of (1) and the function f of (2) is as follows.

$$F(x,t) = \{z : z = f(x,u,t), \ u \in C(x,t)\}$$

THE MODEL

Figure 1 illustrates our simulation. It can be interpreted by two ways.

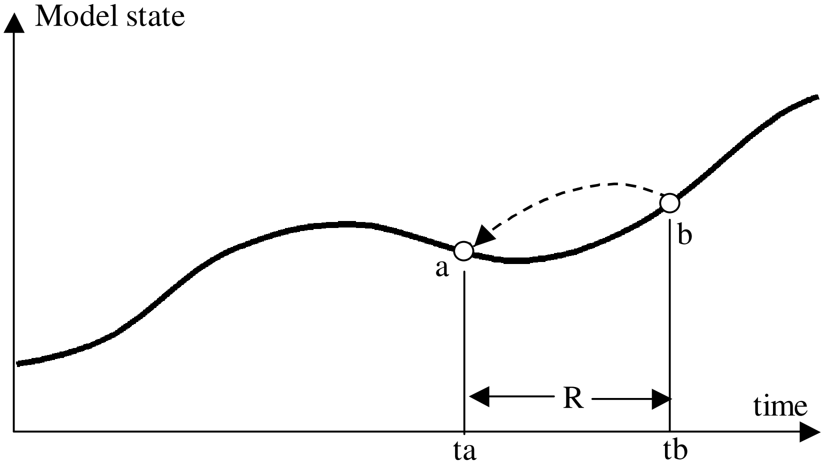

Figure 1. Using the information from the future or traveling to the past.

First, imagine that our model reaches point *a* at time instant *ta*. To advance in time, it needs the information from the future (state *b*). In the science fiction literature this is a situation like "*I go to the future to learn that my marriage is a disaster. So, I decide not to marry at all*". This leads to a paradox.

Second interpretation is as follows. *Being at point b I go back to the past and change the direction of the trajectory of my life*. The same paradox arises. However, the trip to the future does not necessarily lead to a paradox. The following example shows that the use of the uncertain future information may as well result in a solution converging to a unique stable trajectory. The convergence of algorithms similar to that described in this paper can be treated as "stability of time-space in time traveling". This is not science fiction. The problem should be considered as an important (maybe "philosophical") part of modeling theory.

Now, let us restrict to a continuous dynamic ordinary differential equations (ODE) model. To simplify the problem we suppose that the model is autonomous. This means that it does not depend on any external inputs. The model equation is

$$\frac{dx}{dt} = f(x(t), x(t+R), t)$$

Treating the future state $x(t + R)$ as a variable u with uncertain value we can formulate the model as follows

$$
(3) \quad \begin{cases} \dfrac{dx(t)}{dt} = f(x(t), u, t) \\ given \quad x(0) = x_0 \\ u \in C(t+R) \end{cases}
$$

Note that (3) describes a control system (with control u), equivalent to a DI.
The parameter R is positive. The set C represents some restrictions for the values of *u,* depending on the future model state *x(T+R).*
The trajectory of this system cannot be obtained by solving the initial condition Cauchy problem for equation (1), because of the unknown future state *x(t+R)*. One can say that going backward in time we can calculate the trajectory because the prediction converts to time delay. But it is difficult for two reasons. First of all, starting at time instant *T* we still need the information about *x(T+R).* Other question is: From which model state should we start and how to reach the given initial point?

UNCERTAIN FUTURE AND DIFFERENTIAL INCLUSIONS
First, suppose that we have no information on the future at all. In other words, *x(t+R)* is completely uncertain and it can take any value from the whole state space . The equation (3) becomes

$$
(4) \quad \begin{cases} \dfrac{dx(t)}{dt} = f(x(t), C(t+R), t) \\ or \qquad \dfrac{dx(t)}{dt} \in F(x(t), t, R) \\ given \quad x(0) = x_0 \\ f = (f_1, f_2, \ldots, f_n) \\ C \text{ and } F \text{ are sets} \end{cases}
$$

What we get is a *differential inclusion* ***(DI)***, where the right hand side of the equation is a set and not a value. Any function that satisfies equation (4) is a *trajectory* of the DI, but it is not a *solution* to the DI. **The solution to a DI is the reachable set**, defined as the set the graphs of all the model trajectories must belong to. In our case *C(t + R)* is the set of all possible values of *u* (see equation (3)). The right-hand side of the first line of (4) determines the set *F.*
The main point of this article is that the uncertainty of the future can be simulated using differential inclusions. In other words, **the past is the solution to a differential equation, while the future is the solution to a differential inclusion**.
The DIs has been known for about 70 years and there is wide literature available on the DIs theory and applications. The first works have been published in 1931-32 by Marchaud and Zaremba. They used the terms "contingent" or "paratingent" equations. Later, in 1960-70, T. Wazewski and his collaborators published a series of works, referring to the DIs as *orientor conditions* and *orientor fields*. As always occurs with new theories, their works received sever criticism, mainly from some physicists who claimed that it is a stupid way of wasting time while dealing with so abstract an useless theories. Fortunately, the authors

did not abandon the idea and developed the elemental theory of differential inclusions. In the decade 1930-40 such problems as the existence and properties of the solutions to the DIs have been resolved in the finite-dimensional space. After this, many works appear on DIs in more abstract, infinite-dimensional spaces. Within few years after the first publications, the DIs resulted to be the basic tool in the optimal control theory. Recall that optimal trajectories of a dynamic system are those that lay on the boundary of the system reachable set. In the works of Pontiragin, Markus and Lee, Bellman and many others, one of the fundamental problems is the determination of the properties of the reachable sets. Using the theory of Marchaud and Zaremba, T.Wazewski pointed out that in many optimal control problems the resulting control strategy is the so-called *bang-bang* control, generated by switching controllers.

A more extended survey can be found in Raczynski, 1996. One of the best texts on DIs in finite-dimensional as well as abstract spaces is the book of Aubin and Cellina (1984).

DIFFERENTIAL INCLUSION SOLVER

A differential inclusion is a generalization of an ordinary differential equation. In fact, an ODE is a special case of a DI, where the right-hand F is a one-point set. One could expect that a solution algorithm for a DI might be obtained as some extension of known algorithms for the ODEs. Unfortunately, this is not the case. First of all, the solution to a DI is a set and not a function.

One of the properties of the reachable sets is the fact that if a trajectory reaches a point on the boundary of the RS at the final time, then its entire graph must belong to the RS. This fact is well known and used in the optimal control theory. Observe that any trajectory that reaches a point on the boundary of the RS is optimal in some sense. Such trajectories can be calculated using several methods, the main one being the Maximum Principle of Pontriagin. A similar method can be used to construct an algorithm for RS determination. If we can calculate a sufficient number of trajectories that scan the RS boundary, then we can see its shape. The trajectories should be uniformly distributed over the RS boundary. This can be done by some kind of random shooting over the RS boundary. Such shooting has nothing to do with a *simple* or *primitive random shooting*, when the trajectories are generated randomly inside the RS.

My first attempts to develop a DI solver were presented on the IFAC Symposium on Optimization Methods, Varna, 1974. That was a random shooting method, but not a *simple* shooting. That algorithm generated trajectories inside the RS, but the control variable was being modified to obtain a nearly uniform distribution of points inside the RS at the end of the simulated time interval. The DI solver presented here is much more effective.

The DI solver works as follows. The user provides the DI in the form of an equivalent control system. To do it he/she must parameterize the right-hand size (the set F) using an auxiliary variable u. The DI solver automatically generates the equations of so-called conjugated vector $p(t)$ and integrates a set of trajectories, each of them belonging to the boundary of the RS. To achieve this, over each trajectory the Hamiltonian $H(x,p,u,t)$ is maximized. To define the hamiltonian, we must define the conjugated vector $p \in R^n$ that satisfies (by definition) the following equations.

$$(5) \qquad dp_i/dt = -\sum_{j=1}^{n} \frac{\partial f_j}{\partial x_i} p_j + \frac{\partial f_0}{\partial x_i}$$

where $i = 1,..,n$ and f is the vector of the right-hand sides of (2). The hamiltonian is defined as follows.

$$H = \sum_{j=1}^{n} p_j \quad f_j$$

In this case, the Maximum Principle states that the necessary condition for the trajectory to terminate at a boundary point of the reachable set is that the control $u(t)$ maximizes the hamiltonian at each point $t \varepsilon J$. This can be used to generate boundary trajectories of a differential inclusion. If the inclusion is given in the form of a control system (2) we apply the Principle directly. If it is given in the general form (1) we must parametrize the set F and treat the parameter as the control.

This procedure is similar to that used in dynamic optimization. In the optimal control problem the main difficulty consists in the boundary conditions for the state and conjugated vectors. For the sate vector we have the initial conditions given, and for the conjugated vector only the final conditions (at the end of the trajectory) are known, given by the *transversality conditions*. This means that the optimal control algorithm must resolve the corresponding two-point-boundary value problem. In the case of a DI we are in better situation. There is no object function and no transversality conditions. As the consequence, for the vector p we can define the final as well as the initial conditions. Anyway we obtain a trajectory which graph belongs to the RS boundary. Defining initial conditions for p we integrate the trajectory only once, going forward. The only problem is how to generate these initial conditions in order to scan the RS boundary with nearly uniform density. The algorithm is quite simple: the initial conditions for p are generated randomly, due to a density function that is being automatically modified, covering with more density points that correspond to trajectories that fall into a low density regions at the RS boundary. Trajectories that are very close to each other are not stored (storing only one from each eventual cluster). As the result we obtain a set of trajectories covering the RS boundary that can be observed in graphical form and processed. This stochastic part of the algorithm perhaps could be replaced by a deterministic search.

A common error commited while solving similar problems (mainly while treating uncertainty) is to perform a *simple* or *primitive random shooting*. It can be seen (Raczynski.1996) how ineffective is that method. Generating about 10,000 random trajectories with uniform density in the set F we obtain a small cluster that has nothing to do with the true solution. The diameter of this cluster might be only 2-5% of the diameter of the real solution. Using the algorithm proposed here we could obtain the RS boundary with quite good accuracy generating no more than 600 trajectories.

AN APPLICATION TO FUTURE UNCERTAINTY TREATMENT

Now let us go back to our model. Note that the model includes an "ideal predictor" that cannot be replaced by any rough and stochastic one. The idea of the algorithm proposed here is as follows. We start with total uncertainty over the future, and get the solution to the corresponding DI. Then using the reachable set of this DI, we repeat the procedure. The next solution can give us the same reachable set or a smaller one. Note that the new reachable set must be included in the previous one. Repeating these steps several times, we can obtain a very narrow or even one-point set for each time-section of the RS. The shape of this set is the solution to our simulation task.

As a numerical example, consider the following control system shown on figure 2.

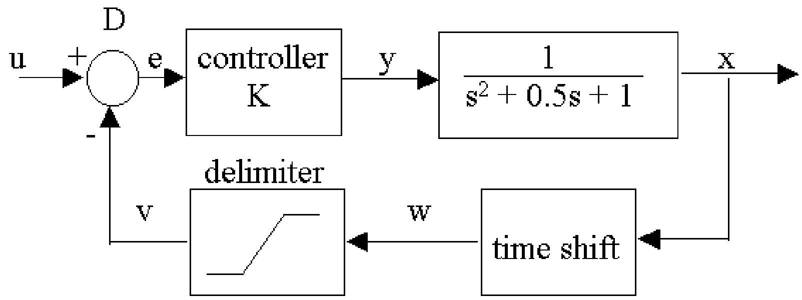

Figure 2. A control system with an ideal predictor.

The goal of the system is to achieve that the output x follows the value of the set point u. The controller is of type P (proportional), the controlled process is a linear dynamic system of the second order, and the delimiter is a simple saturation element. The "time shift" block can represent time delay or time advance. Supposing a time delay we obtain a typical academic control problem described in elementary books on automatic control. Now suppose the instead of time delay we have time advance element, that is, an "ideal predictor". The model equations are as follows.

$$(6)\quad \begin{cases} d^2x/dt^2 + 0.5dx/dt + x(t) = y \\ y = Ke \qquad e = u - v \\ w = x(t+R) \\ v = w \ \mathit{for}\ w < 2 \ \mathit{and}\ w > 0, \\ v = 2 \ \mathit{for}\ w > 2, \quad v = 0 \ \mathit{for}\ w < 0 \\ u = const = 1 \end{cases}$$

Note that the set point is equal to one, so the desired operation point for the model output x is also equal to one. Consequently, the delimiter works in the range 0-2, symmetric with respect to this operation point. Other model parameters are as follows. *K=2, R=0.5,* simulation final time equal to 8.

Introducing two auxiliary variables $x_1 = x$ and $x_2 = dx/dt$ and reordering equation (3) we obtain

$$(7)\quad \begin{cases} \dfrac{dx_1}{dt} = x_2(t) \\ \dfrac{dx_2}{dt} = 2(1-v) - 0.5x_2(t) - x_1(t) \end{cases}$$

In the equations (6) and (7) the value of $x(t+R)$, and, consequently the value of v are uncertain (uncertain future). Replacing v with a set of all its permitted values, we get a differential inclusion.

After introducing the above equations to the DI solver, we obtain the following results.

Iteration 1. In the first iteration we have total uncertainty over the future. This means that $x(t+R)$ can take any value from the whole state space, and v belongs to the set [0,2]. Figure

3 shows the reachable set for the corresponding DI. This image illustrates the region where all possible system trajectories must belong. There is another interpretation of this image. It also depicts the uncertainty of the future when *v* is an external uncertain signal applied to the difference element D (figure 2), which values belong to the set [0,2].

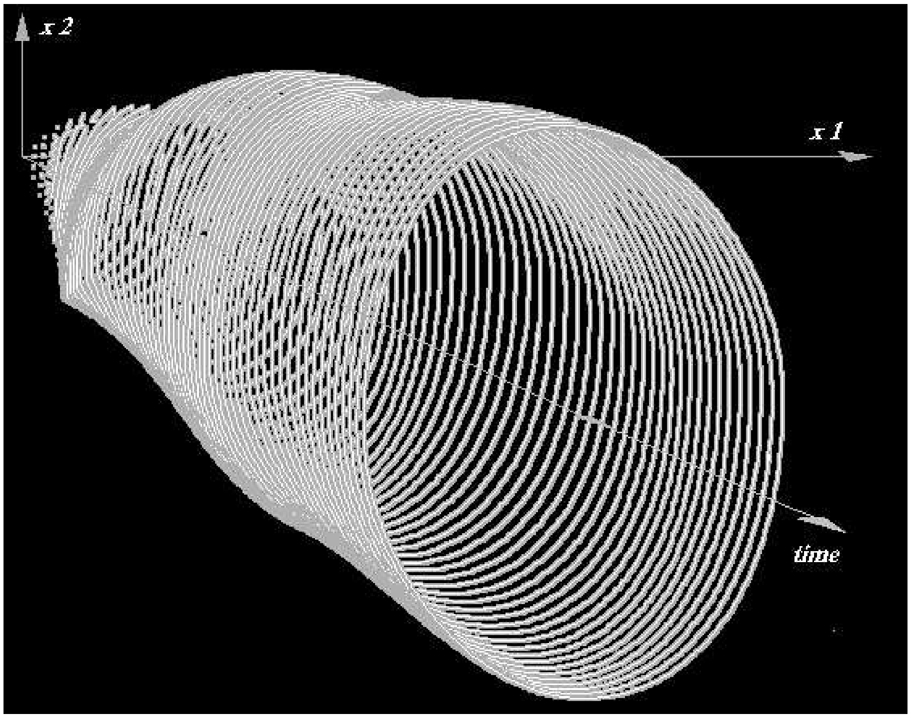

Figure 3. The reachable set for the system trajectories with total uncertainty over the future.

Iterations 2,3,4....

In the iteration *n* we take the reachable set from iteration *n-1* as the restriction for the possible values of *x(t+R)*. This means that *v* now belongs to a subset of [0,2] (it may be the whole set [0,2] again). Figure 4 shows the reachable let after 14 iterations. Observe that nearly a half of the system trajectory is well defined with little uncertainty.

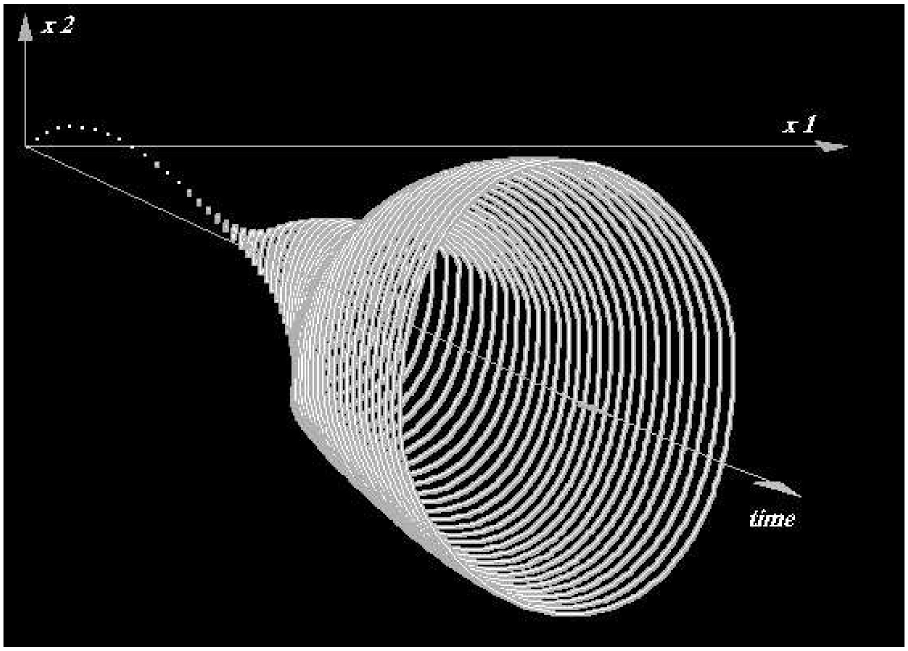

Figure 4. The reachable set for the system trajectories after iteration 14. The future uncertainty is taken from iteration 13.

Next iterations give us better assessments of the system trajectory. Figure 5 shows the reachable set in the iteration 40. Obviously we cannot obtain the whole trajectory. The values of x for the time greater than the simulation final time are still uncertain, so the last interval (final time-R) must remain as a set and not a single line.

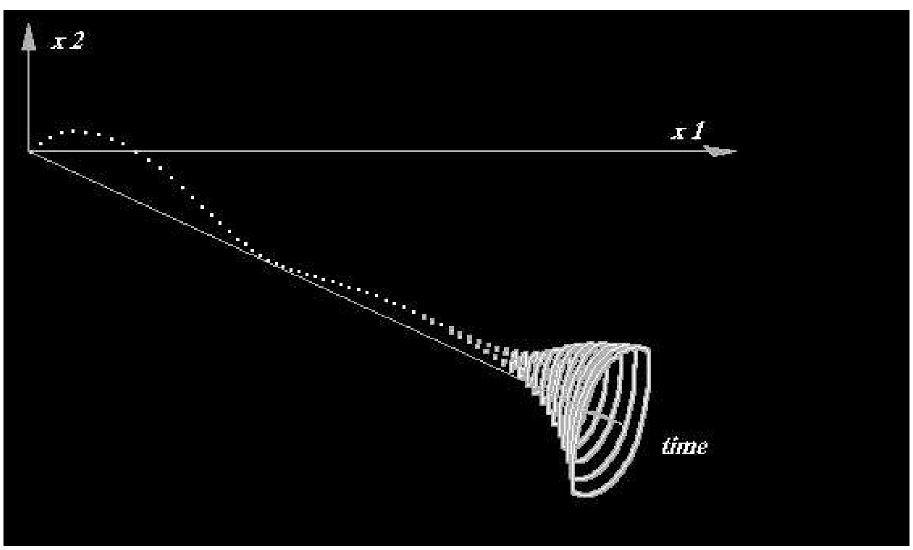

Figure 5. Reachable set (uncertainty) after 40 iterations.

In this experiment and in other similar situations the algorithm converges quite well. However, there are situations when it does not. Its convergence could be a good subject for more theoretical considerations. It can be shown that a small external disturbance added to the feedback signal affects considerably the convergence. It also can be seen that the stability of the model itself is necessary for the convergence. Running the same model with, for example, K=5, the algorithm does not converge to any single trajectory. As for the disturbances in control systems, if we treat them as uncertainty instead of stochastic functions, we again get a DI. Solving such DI we obtain the range for the output variables, that can be useful in such problems as system safety and robustness. Consult the Web page http://www.raczynski.com/pn/uncertainty.htm

CONCLUSIONS

Differential inclusions represent a powerful tool in uncertainty treatment. This may be the uncertainty about the future, like the problem described in this paper, as well as any other kind of uncertainty in dynamic systems. A DI solver exists and works quite well, though further research is necessary.

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